Solving a nonlinear Singular Cauchy Problem of Euler-Poisson-Darboux Equation through Homotopy Perturbation Method

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Abstract
In this paper, we have applied He's homotopy perturbation method (HPM) to solve a nonlinear Singular Cauchy Problem of Euler-Poisson-Darboux Equation. The solution of the problem is much simplified and shorter to arriving at the solution as compared to the technique applied by Carroll and Showalter in the solution of Singular Cauchy Problem. The results are compared with those obtained HPM and exact solutions. The results homotopy perturbation method (HPM) are of high concentration and the method is very effective and succinct.

Keywords: Homotopy Perturbation Method (HPM); Singular Cauchy problem; Partial differential equations.

Introduction
Ji-Huan He proposed a homotopy perturbation method (HPM) based on the use of restricted variations and correction functionals which has found a wide application for the solution of nonlinear ordinary and partial differential equations [1,2]. This method has been widely used to solve linear and nonlinear problems in different fields [3,4].

Until recently, the application of the HPM in nonlinear problems has been developed by scientists and engineers [5], because this method is the most effective and convenient ones for both weakly and strongly non-linear equations. In addition, many authors will apply a kind of analytical technique for nonlinear problems called the homotopy perturbation method (HPM) method to solve approximately the ordinary and partial differential systems and gave great effort to give sophisticated theoretical verification of the homotopy perturbation method [6-8]. The HPM was successfully applied to nonlinear oscillators with discontinuities [3] and bifurcation of nonlinear problems [9]. The analysis derived by homotopy perturbation method (HPM) and Runge–Kutta method. The results are compared with those obtained HPM and Runge–Kutta method in order to verify the accuracy of the proposed method. The Singular Cauchy Problem has been studied since the time of Euler (1770) [10]. Carroll and Showalter dealt primarily with the Cauchy problem for singular and degenerate equation of the form [11]:

\[ A(t)u_t + B(t)u_t + C(t)u = g \]

where \( u(0) \) is a function of \( t \), taking values in a separated locally convex space \( E \), while \( A(t), B(t) \) and \( C(t) \) are families of linear or nonlinear differential type operators acting in \( E \), some of which become zero or infinite at \( t = 0 \). They
considered appropriate initial data \( u(0) \) and \( u(t) \) at \( t=0 \), and \( g \) a suitable \( E \) valued function.

In this article, HPM is used to solve a nonlinear singualr Cauchy problem of Euler-Poisson-Darboux equation:

\[
\frac{\partial^2 u}{\partial t^2} + k \frac{\partial u}{\partial t} - \Delta u = u^n; \text{in} \quad R^n \times (x,0);
\]
\[
u(x,0) = f(x); u_t(x,0) = 0; \text{on} \quad R^n \times \{t=0\}
\]

where \( m>1, k - a \) parameter and \( f(x) \) is smooth with compact support. Assuming \( u \) is at least three degree. On taking the Fourier transform of (1) with respect to \( x=\{x_1, x_2, ..., x_n\} \):

\[
\frac{\partial^2 \tilde{u}}{\partial t^2} + k \frac{\partial \tilde{u}}{\partial t} - \eta \tilde{u} = \tilde{u}^n; \text{in} \quad R^n \times (\eta,0)
\]
\[
u(\eta,0) = f(\eta); u_t(\eta,0) = 0; \text{on} \quad R^n \times \{t=0\}
\]

**Analysis of the homotopy perturbation method**

To explain this method, consider following function:

\[
A(u) - f(r) = 0
\]

With the boundary condition of:

\[
B(u, \frac{\partial u}{\partial n}) = 0
\]

Where \( A(u) \) is defined as follows:

\[
A(u) = u + N(u)
\]

Homotopy perturbation structure is as the following equation:

\[
H(v,p) = (1-p)A(v) + pB(v) = 0
\]

Subjected to the initial condition:

\[
v(\eta,0) = \eta
\]

With \( m=2 \)

Substituting Eq. (8) into Eq. (6) and rearranging based on powers of \( P \)-terms, we can obtain:

\[
p^0 = 0
\]
\[
p^1 \cdot \frac{\partial v_1(\eta,t)}{\partial t} - 2\eta^1 = 0
\]
\[
p^2 \cdot \frac{\partial v_2(\eta,t)}{\partial t} - 2p_2(\eta,t) + k \frac{\partial v_2(\eta,t)}{\partial t} = 0
\]
\[
p^3 = -v_2(\eta,t) \frac{\partial v_1(\eta,t)}{\partial t} - 2p_3(\eta,t) + k \frac{\partial v_3(\eta,t)}{\partial t} = 0
\]

Solving Eqs. (13)-(16), yields:

\[
v_0(\eta,t) = \eta
\]
\[
v_1(\eta,t) = t \eta + 1
\]
\[
v_2(\eta,t) = \frac{1}{6} [t^2 - 4t^2 - 4 \ln(t) + 16 \ln(t) - 6k + 6k \ln(t)]
\]

The solution of Eq. (2) when \( p \rightarrow 1 \) will be as follows:

\[
v = v_0 + pv_1 + p^2v_2
\]
\[ u(\eta, t) = \eta + t \eta (\eta + 1) - \frac{1}{6} t \eta (t^2 \eta^2 - \eta^2 - 16kt^2 \eta - 6k + 6k \ln(\eta)) + \]
\[ \frac{1}{180} t \eta (-40k^2 \eta^2 + 80k^2 \eta^2 + 90k \ln(\eta)^2 - 180 \ln(\eta^2) \eta^2 + 180k^2 \eta^2 + 8\eta t^2 + 24\eta t^4 - 60\eta^2 \ln(\eta) \eta^2 + 15\eta^2 + 180k^2 \eta t^4) \]

The behavior of \( u(\eta, t) \) has been illustrated in Figure 1 and Figure 2 when \( m=2 \).

To solve Eq. (2) by homotopy perturbation method, we construct the following homotopy:
\[ H(v, \eta) = (1 - p) \left( \frac{\partial v}{\partial \xi} - k \frac{\partial v}{\partial t} + |v - \bar{v}| \right) \]
with initial condition:
\[ v(\eta, 0) = \eta \]

Substituting Eq. (8) into Eq. (6) and rearranging based on powers of \( P \)-terms, we can obtain:
\[ p^0 = 0 \]
\[ p^1 = \frac{\partial v}{\partial \xi} (\eta, t) - \eta^2 = 0 \]
\[ p^2 = \frac{\partial^2 v}{\partial \xi^2} (\eta, t) - 3\eta^2 v(\eta, t) + \frac{\partial v}{\partial t} (\eta, t) \]
\[ p^3 = -3\eta^2 v(\eta, t) - 3\eta^2 v(\eta, t) + \frac{\partial^2 v}{\partial \xi^2} (\eta, t) - \frac{\partial v}{\partial t} (\eta, t) \]

Solving Eqs. (24)–(27), yields:
\[ v(\eta, 0) = \eta \]
\[ v(\eta, t) = \frac{1}{2} t \eta (t^2 \eta^2 + t^2 - 2) \]
\[ v(\eta, t) = -\frac{1}{8} t \eta (t^2 \eta^2 - t^2 \eta^2 - 4\eta^2 t + 4k t \eta - 8k + 8k \ln(\eta)) \]

The solution of Eq. (2) when \( p \rightarrow 1 \) will be as follows:
\[ u(\eta, t) = \eta + \frac{1}{2} t \eta (t^2 \eta^2 + t^2 - 2) - \frac{1}{8} t \eta (t^2 \eta^2 - t^2 \eta^2 - 4\eta^2 t + 4k t \eta - 8k + 8k \ln(\eta)) \]
\[ + \frac{1}{240} t \eta (-40k^2 \eta^2 - 40k^2 \eta^2 + 120k^2 \eta^2 + 60k^2 \eta^2 + 60k^2 \eta^2 + 60k^2 \eta^2 + 15k^2 \eta^2 + 120 \ln(\eta) \eta^2 - 120k^2 \ln(\eta)^2 \eta^2 + 240k^2 \eta^2 + 240k^2 \eta^2 + 160k^2 \eta^2 + 36\eta^2 t^4) \]

Let us consider the Eq. (2) when \( m=3 \).

\[ v(\eta, 0) = \eta \]
\[ v(\eta, t) = \frac{1}{2} t \eta (t^2 \eta^2 + t^2 - 2) - \frac{1}{8} t \eta (t^2 \eta^2 - t^2 \eta^2 - 4\eta^2 t + 4k t \eta - 8k + 8k \ln(\eta)) \]
\[ - \frac{1}{240} t \eta (-40k^2 \eta^2 - 40k^2 \eta^2 + 120k^2 \eta^2 + 60k^2 \eta^2 + 60k^2 \eta^2 + 15k^2 \eta^2 + 120 \ln(\eta) \eta^2 - 120k^2 \ln(\eta)^2 \eta^2 + 240k^2 \eta^2 + 240k^2 \eta^2 + 160k^2 \eta^2 + 36\eta^2 t^4) \]
The behavior of $u(\eta, t)$ has been illustrated in Figure 3 and Figure 4 when $m=3$.

Now we consider the Eq. (2) when $m=4$.

To solve Eq. (2) by homotopy perturbation method, we construct the following homotopy:

$$H(v,p) = (1-p) \left[ \frac{\partial^2 u}{\partial t^2} \right] + p \left[ \frac{\partial^2 u}{\partial t^2} + k \frac{\partial u}{\partial t}\right]$$  \hspace{1cm} (33)

with initial condition:

$$v(\eta, 0) = \eta \hspace{1cm} (34)$$

Substituting Eq. (8) into Eq. (6) and rearranging based on powers of $P$-terms, we can obtain:

$$p^3 = 0 \hspace{1cm} (35)$$

$$p^3 = \frac{\partial^2 v_x(\eta, t)}{\partial \eta^2} - 4\eta \frac{\partial^2 v_x(\eta, t)}{\partial \eta^3} = 0 \hspace{1cm} (36)$$

$$p^3 = \frac{\partial^2 v_x(\eta, t)}{\partial \eta^2} - 4\eta \frac{\partial^2 v_x(\eta, t)}{\partial \eta^3} + \frac{k(\frac{\partial v_x(\eta, t)}{\partial \eta})}{t} = 0 \hspace{1cm} (37)$$

$$p^3 = -6\eta \frac{\partial^3 v_x(\eta, t)}{\partial \eta^3} - 4\eta \frac{\partial^3 v_x(\eta, t)}{\partial \eta^4} + \frac{k(\frac{\partial v_x(\eta, t)}{\partial \eta})}{t} = 0 \hspace{1cm} (38)$$

Solving Eqs. (35)–(38), yields:

$$v_x(\eta, t) = \eta \hspace{1cm} (39)$$

$$v_x(\eta, t) = \frac{1}{2} t \eta (t \eta + t + 2) \hspace{1cm} (40)$$

$$v_x(\eta, t) = \frac{1}{6} t \eta (-t \eta^3 - t \eta^3 - 4 t \eta^3 + 3 t \eta^5 + 3 t \eta^2 - 6 k + 6 k \ln(\eta)) \hspace{1cm} (41)$$

$$v_x(\eta, t) = \frac{1}{180} t \eta (90 \eta^2 t^2 + 90 k \eta^2 t - 40 k \eta^2 t - 180 k \ln(t) + 90 k \ln(t) \eta^2 + 160 k \eta^2 t)$$

$$+ 22 \eta^2 t^2 + 9 \eta^2 t^2 + 13 \eta^2 t^2 + 54 \eta^2 t^2 + 180 k^2 + 90 k^2 \eta^2 - 120 k \eta^2 \ln(t) + 78 k \eta^2) \hspace{1cm} (42)$$

The solution of Eq. (2) when $p \to 1$ will be as follows:

$$u(\eta, t) = \eta + \frac{1}{2} t \eta (t \eta + t + 2) \frac{1}{180} t \eta (-t \eta^3 - t \eta^3 - 4 t \eta^3 + 3 t \eta^5 + 3 t \eta^2 - 6 k + 6 k \ln(\eta))$$

$$+ \frac{1}{180} t \eta (90 \eta^2 t^2 + 90 k \eta^2 t - 40 k \eta^2 t - 180 k \ln(t) + 90 k \ln(t) \eta^2 + 160 k \eta^2 t)$$

$$+ 22 \eta^2 t^2 + 9 \eta^2 t^2 + 13 \eta^2 t^2 + 54 \eta^2 t^2 + 180 k^2 + 90 k^2 \eta^2 - 120 k \eta^2 \ln(t) + 78 k \eta^2) \hspace{1cm} (43)$$
The behavior of $u(\eta, t)$ has been illustrated in Figure 5 and Figure 6 when $m=4$.

In the end, we consider the Eq. (2) when $m=5$. To solve Eq. (2) by homotopy perturbation method, we construct the following homotopy:

$$H(v,p) = (1-p) \left[ \frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} + k \frac{\partial^2 u}{\partial t^2} \right] + p \left[ \frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} - p - \frac{\partial u}{\partial t} \right] = 0$$

(44)

with initial condition:

$$v(\eta,0) = \eta$$  

(45)

Substituting Eq. (8) into Eq. (6) and rearranging based on powers of $P$-terms, we can obtain:

$$p^0 = 0$$  

(46)

$$p^1 = \frac{\partial^2 v_1(\eta, t)}{\partial t^2} - \eta^2 = 0$$  

(47)

$$p^2 = \frac{\partial^2 v_2(\eta, t)}{\partial t^2} - 4\eta^2 v_1(\eta, t) + \frac{k}{2} \frac{\partial^2 v_1(\eta, t)}{\partial t} = 0$$  

(48)

$$p^3 = -6\eta^3 v_1(\eta, t)^2 - 4\eta^4 v_1(\eta, t) + \frac{\partial^2 v_2(\eta, t)}{\partial t^2} + \frac{k}{2} \frac{\partial^2 v_1(\eta, t)}{\partial t} = 0$$  

(49)

Solving Eqs. (46)-(49), yields:

$$v_1(\eta, t) = \eta$$  

(50)

$$v_2(\eta, t) = \frac{1}{2} t \eta (t \eta^4 + t \eta + 2)$$  

(51)

$$v_3(\eta, t) = \frac{1}{24} t \eta (t \eta^4 - 5t \eta^3 - 20t \eta^2 + 12k t \eta - 24k + 24k \ln(t))$$  

(52)

The solution of Eq. (2) when $p \rightarrow 1$ will be as follows:

$$u(\eta, t) = \eta + \frac{1}{2} t \eta (t \eta^4 + t \eta + 2) - \frac{1}{24} t \eta (t \eta^4 - 5t \eta^3 - 5t \eta^2 - 20t \eta^2 + 12k t \eta - 24k + 24k \ln(t))$$

(53)

$$+ 102k \eta^4 t^2 + 17k \eta^3 t^2 + 2k t \eta + 12k \eta^2 t - 40k \eta \eta^2 t - 40k \eta^3 t^3 + 72k t \eta^3 t^4 - 40k \eta^4 t^3 + 72k t \eta^4 t^4 - 120k t \eta^4 t^5 + 72k \eta^4 t^6)$$

In the following, we will present the numerical results obtained by applying the Homotopy Perturbation Method (HPM) for $m=4$ and $m=5$.
The compression results for the approximate solution (21)-(32)-(43)-(54) by using the homotopy perturbation method, for the special case m = 2, m = 3, m = 4 and m = 5, are shown in Fig. 9.

**Figure 7.** Comparison between the results obtained by the homotopy perturbation method (HPM) for m = 2, m = 3, m = 4 and m = 5

**Conclusion**

In this paper, an explicit analytical solution is obtained for the singular Cauchy problem by means of the homotopy perturbation method (HPM), which is a powerful mathematical tool in dealing with nonlinear equations. The results clearly indicate the reliability and accuracy of the proposed technique. It is apparently seen that homotopy perturbation method (HPM) is a powerful tool to obtaining exact solutions of partial differential equations or stochastic differential equations arising in various fields of science and engineering and present a rapid convergence for the solutions. The example shows that the results of the present method are in excellent agreement with those obtained by the exact solutions.

**References**

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