

Solving a nonlinear Singular Cauchy Problem of Euler-Poisson-Darboux Equation through Homotopy Perturbation Method

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Abstract

In this paper, we have applied He's homotopy perturbation method (HPM) to solve a nonlinear Singular Cauchy Problem of Euler-Poisson-Darboux Equation. The solution of the problem is much simplified and shorter to arriving at the solution as compared to the technique applied by Carroll and Showalter in the solution of Singular Cauchy Problem. The results are compared with those obtained HPM and exact solutions. The results homotopy perturbation method (HPM) are of high concentration and the method is very effective and succinct.

Keywords: Homotopy Perturbation Method (HPM); Singular Cauchy problem; Partial differential equations.

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1. Introduction

Ji-Huan He proposed a homotopy perturbation method (HPM) based on the use of restricted variations and correction functionals which has found a wide application for the solution of nonlinear ordinary and partial differential equations [1,2]. This method has been widely used to solve linear and nonlinear problems in different fields [3,4].

Until recently, the application of the HPM in nonlinear problems has been developed by scientists and engineers [5], because this method is the most effective and convenient ones for both weakly and strongly non-linear equations. In addition, many authors will apply a kind of analytical technique for nonlinear problems called the homotopy perturbation method (HPM) method to solve approximately the ordinary and partial differential systems and gave great effort to give sophisticated

theoretical verification of the homotopy perturbation method [6-8]. The HPM was successfully applied to nonlinear oscillators with discontinuities [3] and bifurcation of nonlinear problems [9].

The analysis derived by homotopy perturbation method (HPM) and Runge-Kutta method. The results are compared with those obtained HPM and Runge-Kutta method in order to verify the accuracy of the proposed method. The Singular Cauchy Problem has been studied since the time of Euler (1770) [10]. Carroll and Showalter dealt primarily with the Cauchy problem for singular and degenerate equation of the form [11]:

$$A(t)u_{tt} + B(t)u_t + C(t)u = g$$

where $u(0)$ is a function of t , taking values in a separated locally convex space E , while $A(t)$, $B(t)$ and $C(t)$ are families of linear or nonlinear

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differential type operators acting in E , some of which become zero or infinite at $t=0$. They considered appropriate initial data $u(0)$ and $u_t(0)$ at $t=0$, and g a suitable E valued function.

In this article, HPM is used to solve a nonlinear singular Cauchy problem of Euler-Poisson-Darboux equation:

$$\frac{\partial^2 u}{\partial t^2} + \frac{k}{t} \frac{\partial u}{\partial t} - \Delta u = u^m; \text{in } R^n \times (x, 0) \quad (1)$$

$$u(x, 0) = f(x); u_t(x, 0) = 0; \text{on } R^n \times \{t, 0\}$$

Where $m > 1, k - a$ parameter and $f(x)$ is smooth with compact support. Assuming u is at least three degree. On taking the Fourier transform of (1) with respect to $x = (x_1, x_2, \dots, x_n)$;

$$\frac{\partial^2 \tilde{u}}{\partial t^2} + \frac{k}{t} \frac{\partial \tilde{u}}{\partial t} - \eta = \tilde{u}^m; \text{in } R^n \times (\eta, 0) \quad (2)$$

$$u(\eta, 0) = f(\eta); u_t(\eta, 0) = 0; \text{on } R^n \times \{t, 0\}$$

2. Results and Discussion

2.1 Analysis of homotopy perturbation method

To explain this method, consider following function:

$$A(u) - f(r) = 0 \quad (3)$$

With the boundary condition of:

$$B(u, \frac{\partial u}{\partial n}) = 0 \quad (4)$$

Where $A(u)$ is defined as follows:

$$A(u) = L(u) + N(u) \quad (5)$$

Homotopy perturbation structure is as the following equation:

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \quad (6)$$

$$\text{where } v(r, p): \Omega \times [0, 1] \rightarrow R \quad (7)$$

Where $p \in [0, 1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition.

The process of changes in p from zero to unity is that of $u(r)$. We u_0 changing from $v(r, p)$

consider V as following:

$$v_3(\eta, t) = \frac{1}{180} t \eta (-40k\eta^2 t^3 + 80k\eta t^2 + 90k \ln(t)^2 - 180 \ln(t)k^2 + 180k^2 + 8\eta^3 t^5 + 24\eta^2 t^4 - 60\eta t^2 \ln(t)k^2 + 15\eta t^3 + 180k^2 \eta t) \quad (20)$$

$$v = v_0 + p v_1 + p^2 v_2 \quad (8)$$

And the best approximation for equation answer is:

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (9)$$

2.2 Application of the homotopy perturbation method

In this section, we will apply the HPM to solve a nonlinear Singular Cauchy Problem of Euler-Poisson-Darboux Equation. In order to assess the advantages and accuracy of HPM, we consider the equation (2) for different m . According to the HPM (relation (6)), we can construct a homotopy of system as follows:

$$H(v, p) = (1-p) \left[\frac{\partial^2 \tilde{u}}{\partial t^2} \right] + p \left[\frac{\partial^2 \tilde{u}}{\partial t^2} + \frac{k}{t} \frac{\partial \tilde{u}}{\partial t} + |\eta| - \tilde{u}^m \right] \quad (10)$$

Subjected to the initial condition:

$$v(\eta, 0) = \eta \quad (11)$$

With $m=2$

$$H(v, p) = (1-p) \left[\frac{\partial^2 \tilde{u}}{\partial t^2} \right] + p \left[\frac{\partial^2 \tilde{u}}{\partial t^2} + \frac{k}{t} \frac{\partial \tilde{u}}{\partial t} + |\eta| - \tilde{u}^2 \right] \quad (12)$$

Substituting Eq. (8) into Eq. (6) and rearranging based on powers of P -terms, we can obtain:

$$p^0 = 0 \quad (13)$$

$$p^1 = \frac{\partial^2 v_1(\eta, t)}{\partial t^2} - 2\eta^2 = 0 \quad (14)$$

$$p^2 = \frac{\partial^2 v_2(\eta, t)}{\partial t^2} - 2\eta v_1(\eta, t) + \frac{k \left(\frac{\partial v_1(\eta, t)}{\partial t} \right)}{t} = 0 \quad (15)$$

$$p^3 = -v_1(\eta, t)^2 - 2\eta v_2(\eta, t) + \frac{\partial^2 v_3(\eta, t)}{\partial t^2} + \frac{k \left(\frac{\partial v_2(\eta, t)}{\partial t} \right)}{t} = 0 \quad (16)$$

Solving Eqs. (13)-(16), yields:

$$v_0(\eta, t) = \eta \quad (17)$$

$$v_1(\eta, t) = t\eta(t\eta + 1) \quad (18)$$

$$v_2(\eta, t) = -\frac{1}{6} t \eta (-t^3 \eta^2 - \eta t^2 + 16kt\eta - 6k + 6k \ln(t)) \quad (19)$$

The solution of Eq. (2) when $p \rightarrow 1$ will be as follows:

$$u(\eta, t) = \eta + t\eta(t\eta + 1) - \frac{1}{6}t\eta(-t^3\eta^2 - \eta t^2 + 16kt\eta - 6k + 6k \ln(t)) + \frac{1}{180}t\eta(-40k\eta^2t^3 + 80k\eta t^2 + 90k \ln(t)^2 - 180 \ln(t)k^2 + 180k^2 + 8\eta^3t^5 + 24\eta^2t^4 - 60\eta t^2 \ln(t)k^2 + 15\eta t^3 + 180k^2\eta t) \quad (21)$$

The behavior of $u(\eta, t)$ has been illustrated in Figure 1 and Figure 2 when $m=2$.

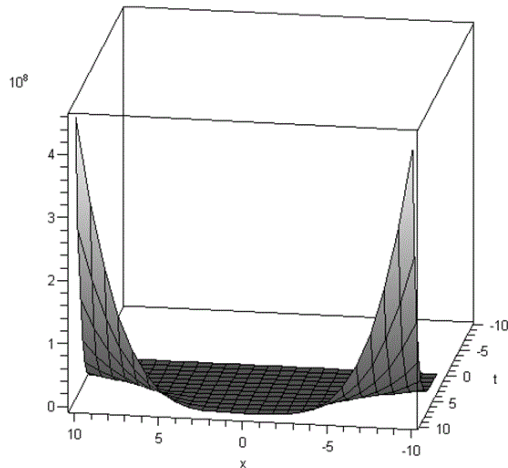


Figure 1. The surface generated from $u(\eta, t)$ of Homotopy Perturbation Method (HPM) for $m=2$ when $k = 1$

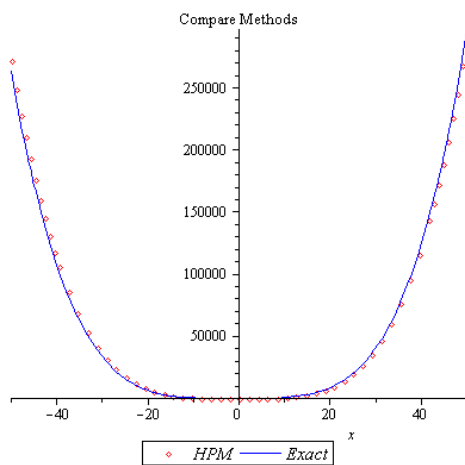


Figure 2. Comparison between results of the HPM with exact solution at $k = 1$ and $t = 1$ Let us we consider the Eq. (2) when $m=3$.

$$v_3(\eta, t) = \frac{1}{240}t\eta(-40k\eta^4t^3 - 40k\eta^3t^3 + 120k^2\eta^2t + 120k^2\eta t + 60\eta^2t^3 + 6\eta^4t^5 + 15\eta^5t^5 + 9\eta^6t^5 + 120 \ln(t)^2k^2 - 120k\eta^2t^2 \ln(t) + 54\eta^4t^4 + 240k^2 - 240k^2 \ln(t) + 160k\eta^2t^2 + 36\eta^3t^4) \quad (31)$$

The solution of Eq. (2) when $p \rightarrow 1$ will be as follows:

$$u(\eta, t) = \eta + \frac{1}{2}t\eta(t\eta^2 + t\eta + 2) - \frac{1}{8}t\eta(-t^3\eta^4 - t^3\eta^3 - 4\eta^2tk + 4kt\eta - 8k + 8k \ln(t)) + \frac{1}{240}t\eta(-40k\eta^4t^3 - 40k\eta^3t^3 + 120k^2\eta^2t + 120k^2\eta t + 60\eta^2t^3 + 6\eta^4t^5 + 15\eta^5t^5 + 9\eta^6t^5 + 120 \ln(t)^2k^2 - 120k\eta^2t^2 \ln(t) + 54\eta^4t^4 + 240k^2 - 240k^2 \ln(t) + 160k\eta^2t^2 + 36\eta^3t^4) \quad (32)$$

To solve Eq. (2) by homotopy perturbation method, we construct the following homotopy:

$$H(v, p) = (1-p) \left[\frac{\partial^2 \bar{u}}{\partial t^2} \right] + p \left[\frac{\partial^2 \bar{u}}{\partial t^2} + \frac{k}{t} \frac{\partial \bar{u}}{\partial t} + |\eta| - \bar{u}^3 \right] \quad (22)$$

with initial condition:

$$v(\eta, 0) = \eta \quad (23)$$

Substituting Eq. (8) into Eq. (6) and rearranging based on powers of P -terms, we can obtain:

$$p^0 = 0 \quad (24)$$

$$p^1 = \frac{\partial^2 v_1(\eta, t)}{\partial t^2} - \eta^2 - \eta^3 = 0 \quad (25)$$

$$p^2 = \frac{\partial^2 v_2(\eta, t)}{\partial t^2} - 3\eta^2 v_1(\eta, t) + \frac{k \left(\frac{\partial v_1(\eta, t)}{\partial t} \right)}{t} = 0 \quad (26)$$

$$p^3 = -3\eta v_1(\eta, t)^2 - 3\eta^2 v_2(\eta, t) + \frac{\partial^2 v_3(\eta, t)}{\partial t^2} + \frac{k \left(\frac{\partial v_2(\eta, t)}{\partial t} \right)}{t} = 0 \quad (27)$$

Solving Eqs. (24)–(27), yields:

$$v_0(\eta, t) = \eta \quad (28)$$

$$v_1(\eta, t) = \frac{1}{2}t\eta(t\eta^2 + t\eta + 2) \quad (29)$$

$$v_2(\eta, t) = -\frac{1}{8}t\eta(-t^3\eta^4 - t^3\eta^3 - 4\eta^2tk + 4kt\eta - 8k + 8k \ln(t)) \quad (30)$$

The behavior of $u(\eta, t)$ has been illustrated in Figure 3 and Figure 4 when $m=3$

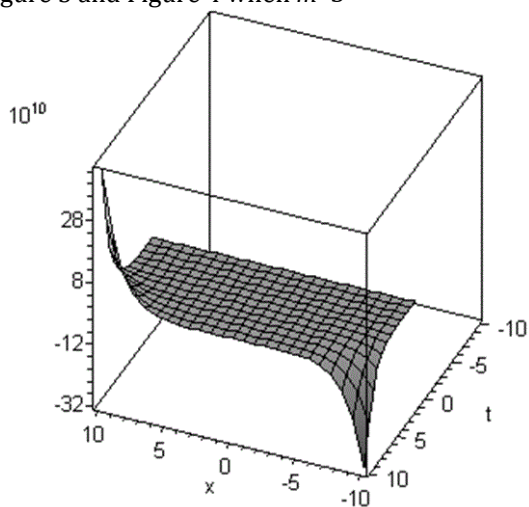


Figure 3. The surface generated from $u(\eta, t)$ of Homotopy Perturbation Method (HPM) for $m=3$ when $k = 1$

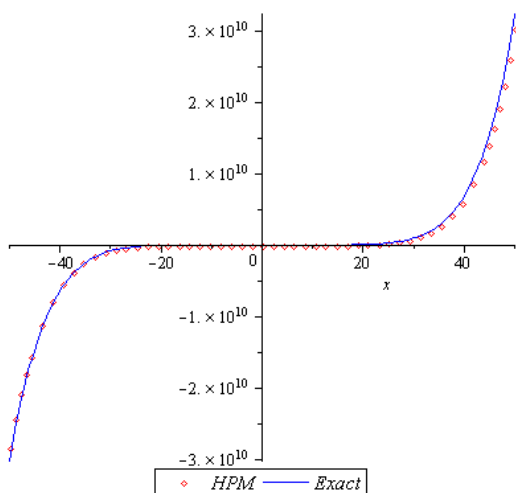


Figure 4. Comparison between results of the HPM with exact solution at $k = 1$ and $t = 1$

Now we consider the Eq. (2) when $m=4$. To solve Eq. (2) by homotopy perturbation method, we construct the following homotopy:

$$H(v, p) = (1-p) \left[\frac{\partial^2 \bar{u}}{\partial t^2} \right] + p \left[\frac{\partial^2 \bar{u}}{\partial t^2} + \frac{k}{t} \frac{\partial \bar{u}}{\partial t} + |\eta| - \bar{u}^4 \right] \quad (33)$$

with initial condition:

$$v(\eta, 0) = \eta \quad (34)$$

Substituting Eq. (8) into Eq. (6) and rearranging based on powers of P -terms, we can obtain:

$$p^0 = 0 \quad (35)$$

$$p^1 = \frac{\partial^2 v_1(\eta, t)}{\partial t^2} - \eta^2 - \eta^4 = 0 \quad (36)$$

$$p^2 = \frac{\partial^2 v_2(\eta, t)}{\partial t^2} - 4\eta^3 v_1(\eta, t) + \frac{k \left(\frac{\partial v_1(\eta, t)}{\partial t} \right)}{t} = 0 \quad (37)$$

$$p^3 = -6\eta^2 v_1(\eta, t)^2 - 4\eta^3 v_2(\eta, t) + \frac{\partial^2 v_3(\eta, t)}{\partial t^2} + \frac{k \left(\frac{\partial v_2(\eta, t)}{\partial t} \right)}{t} = 0 \quad (38)$$

Solving Eqs. (35)–(38), yields:

$$v_0(\eta, t) = \eta \quad (39)$$

$$v_1(\eta, t) = \frac{1}{2} t \eta (t \eta^3 + t \eta + 2) \quad (40)$$

$$v_2(\eta, t) = -\frac{1}{6} t \eta (-t^3 \eta^6 - t^3 \eta^4 - 4\eta^3 t^2 + 3k t \eta^3 + 3k t \eta - 6k + 6k \ln(t)) \quad (41)$$

$$v_3(\eta, t) = \frac{1}{180} t \eta (90\eta^3 t^3 + 90k^2 \eta^3 t - 40k \eta^4 t^3 - 180k^2 \ln(t) + 90 \ln(t)^2 k^2 + 160k \eta^3 t^2 + 22\eta^7 t^5 + 9\eta^5 t^5 + 13\eta^9 t^5 + 54\eta^4 t^4 + 180k^2 + 90k^2 \eta t - 120k \eta^3 t^2 \ln(t) + 78\eta^6 t^4) \quad (42)$$

The solution of Eq. (2) when $p \rightarrow 1$ will be as follows:

$$u(\eta, t) = \eta + \frac{1}{2} t \eta (t \eta^3 + t \eta + 2) - \frac{1}{6} t \eta (-t^3 \eta^6 - t^3 \eta^4 - 4\eta^3 t^2 + 3k t \eta^3 + 3k t \eta - 6k + 6k \ln(t)) + \frac{1}{180} t \eta (90\eta^3 t^3 + 90k^2 \eta^3 t - 40k \eta^4 t^3 - 180k^2 \ln(t) + 90 \ln(t)^2 k^2 + 160k \eta^3 t^2 + 22\eta^7 t^5 + 9\eta^5 t^5 + 13\eta^9 t^5 + 54\eta^4 t^4 + 180k^2 + 90k^2 \eta t - 120k \eta^3 t^2 \ln(t) + 78\eta^6 t^4) \quad (43)$$

The behavior of $u(\eta, t)$ has been illustrated in Figure 5 and Figure 6 when $m=4$

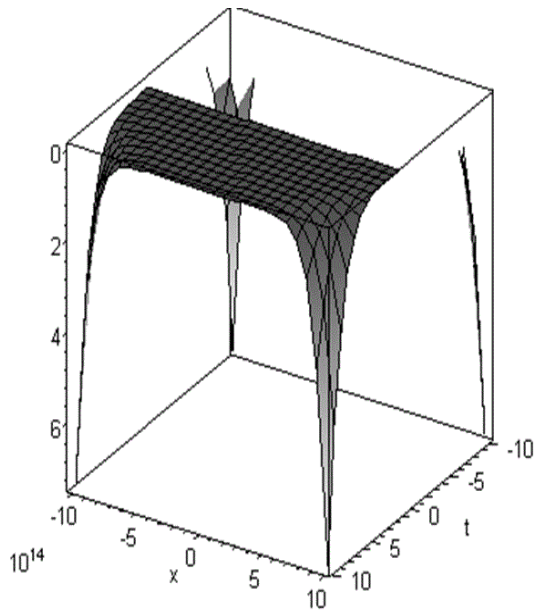


Figure 5. The surface generated from $u(\eta, t)$ of Homotopy Perturbation Method (HPM) for $m=4$ when $k = 1$

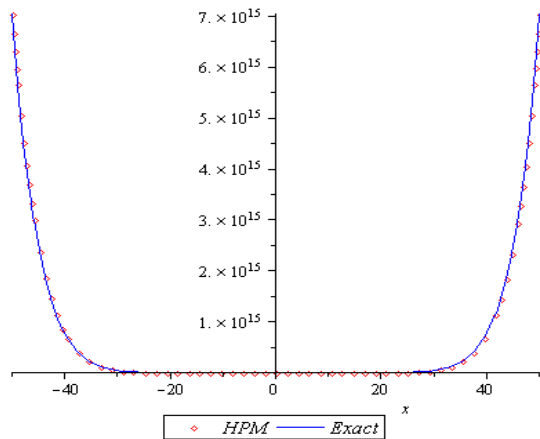


Figure 6. Comparison between results of the HPM with exact solution at $k = 1$ and $t = 1$

$$v_3(\eta, t) = \frac{1}{144} t \eta (72k^2 \ln(t)^2 - 144k^2 \ln(t) + 160k\eta^4 t^2 + 72\eta^4 t k^2 + 72\eta t k^2 + 144k^2 + 102\eta^8 t^4 + 17\eta^{12} t^5 + 29\eta^9 t^5 + 12\eta^6 t^5 + 120\eta^4 t^3 - 40k\eta^8 t^3 - 40k\eta^5 t^3 + 72\eta^5 t^4 - 40k\eta^8 t^3 - 40k\eta^5 t^3 + 72\eta^5 t^4 - 120k\eta^4 t^2 \ln(t)) \quad (53)$$

The solution of Eq. (2) when $p \rightarrow 1$ will be as follows:

$$u(\eta, t) = \eta + \frac{1}{2} t \eta (t \eta^4 + t \eta + 2) - \frac{1}{24} t \eta (-5t^3 \eta^8 - 5t^3 \eta^5 - 20\eta^4 t^2 + 12k t \eta^4 + 12k t \eta - 24k + 24k \ln(t) + \frac{1}{144} t \eta (72k^2 \ln(t)^2 - 144k^2 \ln(t) + 160k\eta^4 t^2 + 72\eta^4 t k^2 + 72\eta t k^2 + 144k^2 + 102\eta^8 t^4 + 17\eta^{12} t^5 + 29\eta^9 t^5 + 12\eta^6 t^5 + 120\eta^4 t^3 - 40k\eta^8 t^3 - 40k\eta^5 t^3 + 72\eta^5 t^4 - 40k\eta^8 t^3 - 40k\eta^5 t^3 + 72\eta^5 t^4 - 120k\eta^4 t^2 \ln(t))) \quad (54)$$

In the end, we consider the Eq. (2) when $m=5$. To solve Eq. (2) by homotopy perturbation method, we construct the following homotopy:

$$H(v, p) = (1-p) \left[\frac{\partial^2 \tilde{u}}{\partial t^2} \right] + p \left[\frac{\partial^2 \tilde{u}}{\partial t^2} + \frac{k}{t} \frac{\partial \tilde{u}}{\partial t} + |\eta| - \tilde{u}^5 \right] \quad (44)$$

with initial condition:

$$v(\eta, 0) = \eta \quad (45)$$

Substituting Eq. (8) into Eq. (6) and rearranging based on powers of P -terms, we can obtain:

$$p^0 = 0 \quad (46)$$

$$p^1 = \frac{\partial^2 v_1(\eta, t)}{\partial t^2} - \eta^2 - \eta^4 = 0 \quad (47)$$

$$p^2 = \frac{\partial^2 v_2(\eta, t)}{\partial t^2} - 4\eta^3 v_1(\eta, t) + \frac{k \left(\frac{\partial v_1(\eta, t)}{\partial t} \right)}{t} = 0 \quad (48)$$

$$p^3 = -6\eta^2 v_1(\eta, t)^2 - 4\eta^3 v_2(\eta, t) + \frac{\partial^2 v_3(\eta, t)}{\partial t^2} + \frac{k \left(\frac{\partial v_2(\eta, t)}{\partial t} \right)}{t} = 0 \quad (49)$$

Solving Eqs. (46)–(49), yields:

$$v_0(\eta, t) = \eta \quad (50)$$

$$v_1(\eta, t) = \frac{1}{2} t \eta (t \eta^4 + t \eta + 2) \quad (51)$$

$$v_2(\eta, t) = -\frac{1}{24} t \eta (-5t^3 \eta^8 - 5t^3 \eta^5 - 20\eta^4 t^2 + 12k t \eta^4 + 12k t \eta - 24k + 24k \ln(t)) \quad (52)$$

The compression results for the approximate solution (21)-(32)-(43)-(54) by using the homotopy perturbation method, for the special case $m=2$, $m=3$, $m=4$ and $m=5$, are shown in Fig. 9. Figure 7 shows Comparison between the results obtained by the homotopy perturbation method for $m=2$, $m=3$, $m=4$ and $m=5$.

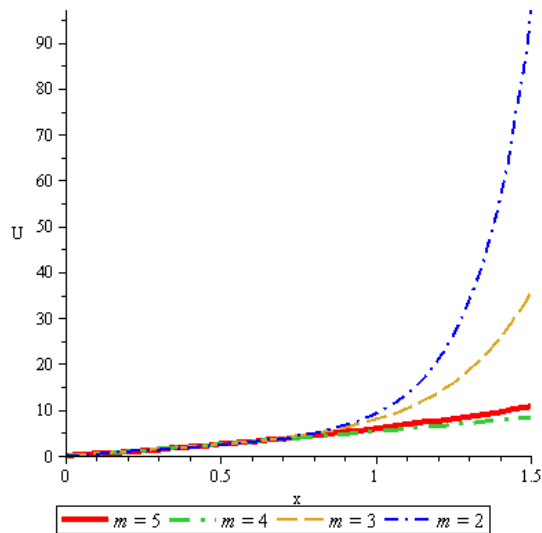


Figure 7. Comparison between the results obtained by the homotopy perturbation method (HPM) for $m=2$, $m=3$, $m=4$ and $m=5$

3. Conclusion

In this paper, an explicit analytical solution is obtained for the singular Cauchy problem by means of the homotopy perturbation method (HPM), which is a powerful mathematical tool in dealing with nonlinear equations. The results clearly indicate the reliability and accuracy of the proposed technique. It is apparently seen that homotopy perturbation method (HPM) is a powerful tool to obtaining exact solutions of partial differential equations or stochastic differential equations arising in various fields of science and engineering and present a rapid convergence for the solutions. The example shows that the results of the present

method are in excellent agreement with those obtained by the exact solutions.

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