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Research Paper

An Efficient Technique in Finding the Exact Solutions for Cauchy Problems

S. Saeedi[1], M. Mighani[2*]

[1]Department of Architectural Engineering, Shahid Beheshti University, Tehran, Iran
[2]Department of Architectural Engineering, Shomal University, Amol, Iran

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Abstract

In this paper, the Reconstruction of Variational Iteration Method (RVIM) is used to solve the Cauchy problem. The new applied algorithm is a powerful and efficient technique in finding solutions for the linear and nonlinear equations using only few terms. The Reconstruction of Variational Iteration Method(RVIM) technique is independent of any small parameters at all. Besides, it provides us with a simple way to ensure the convergence of solution series, so that we can always get accurate enough approximations. Some examples are given to elucidate the solution procedure and reliability of the obtained results.

Keywords: Reconstruction of Variational Iteration Method; Cauchy Problem, Inviscid Burgers' Equation, Transport Equation

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1. Introduction

Nonlinear phenomena play a crucial role in applied mathematics and engineering; hence solving of governing equations have been one of the most time-consuming and difficult affairs among researchers. Therefore, many researchers and scientist of mathematics have recently paid much attention to find and develop approximate solutions. If there is no small parameter in the equation, the traditional perturbation methods cannot be applied directly. Recently, considerable attention has been directed towards the analytical solutions for nonlinear equations without possible small parameters. The traditional perturbation methods have many shortcomings, and they are not valid for strongly nonlinear equations. To overcome the shortcomings, many new techniques have appeared in open literature [1-4]. Variational methods have been, and continue to be, popular tools for nonlinear analysis. When contrasted with other approximate analytical methods, variational methods combine the following two advantages: (1) they provide physical insight into the nature of the solution of the problem; (2) the obtained solutions are the best among all the possible trial-functions.

Variational iteration method is based on a general Lagrange multiplier, and it can be applied to various nonlinear equations. In 2009 Hesameddini and Latifizadeh [5], proposed a new method based on Laplace transform -Reconstruction of variational iteration method (RVIM) in which the correctional function of the variational iteration method is obtained without using the Variational Theory. Therefore, in this method the complexity in calculating the Lagrange multiplier has been removed.

In this paper, we use the method to discuss the first-order partial differential equation in the form [6] (McOwen, 1996):

* Corresponding Author Email: mighani.mona@yahoo.com

$$u_t(x.t) + a(x.t)u_x(x.t) = \psi(x). \quad x \in R.t > 0 \tag{1}$$

$$u(x.0) = \varphi(x). \quad x \in R. \tag{2}$$

When $a(x.t) = a$ is a constant and $\psi(x) = 0$ Eq. (1) is a linear equation called the transport equation which can describe many interesting phenomena such as the spread of AIDS, the moving of wind. When $a(x.t) = u(x.t)$ the equation is called the inviscid Burgers' equation arising in one-dimensional stream of particles or fluid having zero viscosity.

2. Description of the method

To clarify the basic ideas of our proposed method in [7-9], we consider the following differential equation same as VIM based on Lagrange multiplier [10]:

$$Lu(x_1.\cdots.x_k) + Nu(x_1.\cdots.x_k) = f(x_1.\cdots.x_k) \tag{3}$$

By suppose that

$$Lu(x_1.\cdots.x_k) = \sum_{i=0}^{k} L_{x_i}u(x_i) \tag{4}$$

where L is a linear operator, N a nonlinear operator and $f(x_1.\cdots.x_k)$ an inhomogeneous term.

we can rewrite equation (3) down a correction functional as follows:

$$L_{x_j}u(x_j) = \underbrace{f(x_1.\cdots.x_{kn}) - Nu(x_1.\cdots.x_k) - \sum_{\substack{i=0\\ i\neq j}}^{k} L_{x_i}u(x_i)}_{h((x_1.\cdots.x_k).u(x_1.\cdots.x_k))} \tag{5}$$

Therefore

$$L_{x_j}u(x_j) = h\big((x_1.\cdots.x_k).u(x_1.\cdots.x_k)\big) \tag{6}$$

With artificial initial conditions being zero regarding the independent variable x_j .

By taking Laplace transform of both sides of the equation (6) in the usual way and using the artificial initial conditions, we obtain the result as follows:

$$P(s).U(x_1.\cdots.x_{i-1}.s.x_{i+1}.x_k) = H\big((x_1.\cdots.x_{i-1}.s.x_{i+1}.x_k).u\big) \tag{7}$$

Where P(s) is a polynomial with the degree of the highest derivative in equation (7), (the same as the highest order of the linear operator L_{x_j}). The following relations are possible;

$$\mathcal{L}[h] = H \tag{8-a}$$

$$B(s) = \frac{1}{P(s)} \tag{8-b}$$

$$\mathcal{L}[b(x_i)] = B(s) \tag{8-c}$$

Which that in equation (8-a) the function $H\big((x_1.\cdots.x_{i-1}.s.x_{i+1}.x_k).u\big)$ and $.h\big((x_1.\cdots.x_{i-1}.x_i.x_{i+1}.x_k).u\big)$ have been abbreviated as $H.h$ respectively. Hence, rewrite the equation (7) as;

$$U(x_1.\cdots.x_{i-1}.s.x_{i+1}.x_k) = H\big((x_1.\cdots.x_{i-1}.s.x_{i+1}.x_k).u\big).B(s) \tag{9}$$

Now, by applying the inverse Laplace Transform on both sides of equation (9) and by using the (8-a) - (8-c), we have:

$$u(x_1.\cdots.x_{i-1}.x_i.x_{i+1}.x_k) = \int_0^{x_i} h\big((x_1.\cdots.x_{i-1}.\tau.x_{i+1}.x_k).u\big).b(x_i - \tau)d\tau \tag{10}$$

Now, we must impose the actual initial conditions to obtain the solution of the equation (3). Thus, we have the following iteration formulation:

$$u_{n+1}(x_1.\cdots.x_{i-1}.x_i.x_{i+1}.x_k) = u_0(x_1.\cdots.x_{i-1}.x_i.x_{i+1}.x_k) + \int_0^{x_i} \{h\big((x_1.\cdots.x_{i-1}.\tau.x_{i+1}.x_k).u_n\big).b(x_i - \tau)\}d\tau \tag{11}$$

where u_0 is initial solution with or without unknown parameters. Assuming u_0 is the solution of ¸Lu, with initial/boundary conditions of the main problem, In case of no unknown parameters, u_0 should satisfy initial/ boundary conditions. When some unknown parameters are involved in u_0, the unknown parameters can be identified by initial/boundary conditions after few iterations, this technology is very effective in dealing with boundary problems. It is worth mentioning that, in fact, the Lagrange multiplier in the He's variational iteration method is $\lambda(\tau) = b(x_i - \tau)$ as shown in [5].

The initial values are usually used for selecting the zeroth approximationu_0. With u_0 determined, then several approximations u_n $n > 0$, follow immediately. Consequently, the exact solution may be obtained by using

$$u(x_1.\cdots.x_{i-1}.x_i.x_{i+1}.x_k) = \lim_{n\to\infty} u_n(x_1.\cdots.x_{i-1}.x_i.x_{i+1}.x_k). \tag{12}$$

In what follows, we will apply the RVIM method to homogeneous/non-homogeneous parabolic partial differential equations to illustrate the strength of the method and to establish exact solutions for these problems.

3. Application

Since our focus is on the ideas and basic principles, we shall consider only the simplest possible equations to clearly illustrate the solution procedure. In particular, we will focus on pure Cauchy problems. These problems are initial value problems.

3.1 Example1

Consider the transport equation [6]:

$$u_t(x.t) + au_x(x.t) = 0. x \in R \ .t > 0 \quad (13)$$

Subject to the initial conditions:

$$u(x.0) = x^2. \quad x \in R \quad (14)$$

By selection of auxiliary linear operator, so the Eq. (13) is rewritten as

$$L_t u(x.t) = \frac{\partial u}{\partial t} = \overbrace{-\left(a\frac{\partial u}{\partial t}\right)}^{h(x.\xi.u)} \quad (15)$$

Now Laplace transform is implemented with respect to independent variable x on both sides of eq. (15) and by using the new artificial initial condition (which all of them are zero) we have:

$$s \cup (x.t) = \mathbb{L}\{h(x.t.u)\} \quad (16)$$

$$\cup (x.t) = \frac{\mathbb{L}\{h(x.t.u)\}}{s} \quad (17)$$

And whereas Laplace inverse transform of 1/s is as follows

$$\mathbb{L}^{-1}[{}^1/_s] = \quad (18)$$

$$u(x.t) = \int_0^t h(x.\xi.u)d\xi \quad (19)$$

Therefore, by using the Eq. (10), (11) one can obtain the following RVIM's iteration formula in the t-direction

$$u_{n+1}(x.t) = u_0(x.t) + \int_0^t h(x.\xi.u_n)d\xi \quad (20)$$

Now we start with an arbitrary initial approximation $u(x.0) = x^2$. that satisfies the initial condition and by using the RVIM iteration formula (20), we have the following successive approximation

$$u_1(x.t) = x^2 - 2atx. \quad (21)$$

$$u_2(x.t) = x^2 - 2atx + a^2t^2 \quad (22)$$

$$u_3(x.t) = x^2 - 2atx + a^2t^2 \quad (23)$$

$$\vdots$$

Whereas, the RVIM method admits the use of $u = \lim_{n\to\infty} u_n$,

which gives the exact solution

$$u(x.t) = x^2 - 2atx + a^2t^2 \quad (24)$$

3.2 Example 2

Consider the nonlinear Cauchy problem [11]:

$$u_t(x.t) + xu_x(x.t) = 0. x \in R \ .t > 0 \quad (25)$$

Subject to the initial conditions:

$$u(x.0) = x^2. \quad x \in R \quad (26)$$

Such as previous examples, for implementation of the RVIM technique, first of all we need to choose the auxiliary linear operator as

$$L_t u(x.t) = \frac{\partial u}{\partial t} = \overbrace{-\left(x\frac{\partial u}{\partial t}\right)}^{h(x.\xi.u)} \quad (27)$$

Accordingly, after taking Laplace Transform by using the artificial initial condition as in [5], on both side of Eq. (25), the following RVIM iteration formula in the t-direction can be obtained

$$u_{n+1}(x.t) = u_0(x.t) - \int_0^t x\frac{\partial u_n}{\partial x}(x.\xi)d\xi \quad (28)$$

By the RVIM'S recurrent formula in Eq. (28), the terms of the sequence $\{u_n\}$ are constructed as follows, so that we choose its initial approximate solution as $u_0(x.t) = x^2$.

$$u_1(x.t) = x^2 - 2tx^2 \quad (29)$$

$$u_2(x.t) = x^2 - 2tx^2 + 2t^2x^2 \quad (30)$$

$$u_3(x.t) = x^2 - 2tx^2 + 2t^2x^2 - \frac{4}{3}t^3x^2 \quad (31)$$

The next terms of $\{u_n\}$ can be determined in a similar way and we can construct the approximation of as

$$u_n = x^2e^{-2t} \quad (32)$$

The approximation obtained by RVIM procedure converges to the exact solution.

3.3 Example 3

Now we solve the following non-homogeneous Cauchy problem [11]:

$$u_t(x.t) + u_x(x.t) = x. \quad x \in R \ .t > 0$$
$$u(x.0) = e^x. \quad x \in R. \quad (33)$$

To apply RVIM to this equation with initial and boundary conditions, according to (5) and (6), we have:

$$L_t u(x.t) = \frac{\partial u}{\partial t} = \overbrace{\left(-\frac{\partial u}{\partial x} + x\right)}^{h(x.\xi.u)} \quad (34)$$

Therefore, as previous Examples the RVIM iterative formula can be expressed as:

$$u_{n+1}(x.t) = u_0(x.t) + \int_0^t (-u_{n_x}(x.\xi) + x)d\,\xi \quad (35)$$

With the aid of the initial approximation $u_0(x.t) = e^x$ and using the RVIM iteration, we can obtain directly the rest of the other components as follows

$$u_1(x.t) = e^x - te^x + tx \quad (36)$$

$$u_2(x.t) = e^x - te^x + tx - \frac{t^2}{2} + e^x\frac{t^2}{2} \quad (37)$$

$$u_3(x.t) = e^x - te^x + tx - \frac{t^2}{2} + e^x\frac{t^2}{2} - e^x\frac{t^6}{6} \quad (38)$$

The RVIM admits the use of $u = \lim_{n\to\infty} u_n$. which gives the exact solution

$$u(x.t) = t\left(x - \frac{t}{2}\right) + e^{x-t} \quad (39)$$

3.4 Example4

Consider the inviscid Burgers' equation (Tveito and Winther, 2005):

$$\begin{aligned} &u_t(x.t) + u(x.t)u_x(x.t) = 0. \quad x \in R\,.\mathrm{t} > 0 \\ &u(x.0) = x. \quad x \in R. \end{aligned} \quad (40)$$

To implementation of the RVIM method to this differential equation with initial and boundary conditions, according to (5) and (6), the auxiliary linear operator is selected as

$$L_t u(x.t) = \frac{\partial^2 u}{\partial t^2} = \overbrace{\left(-\frac{\partial u}{\partial x}u\right)}^{h(x.\xi.u)} \quad (41)$$

Therefore, as previous Examples the RVIM iterative formula can be expressed as:

$$u_{n+1}(x.t) = u_0(x.t) + \int_0^t u_n(x.\xi)u_{n_x}(x.\xi)d\xi \quad (42)$$

We start with an initial approximation $\mathrm{u_0(x.t) = x}$; by the iteration formula (42), we can obtain the first few components as follows

$$u_1(x.t) = x - tx \quad (43)$$

$$u_2(x.t) = x - tx + t^2x - \frac{t^3x}{3} \quad (44)$$

$$u_3(x.t) = x - tx + t^2x - t^3x + \frac{2t^4x}{3} - \frac{2t^5x}{3} + \frac{2t^6x}{9} - \frac{t^7x}{63} \quad (45)$$

And the rest of the components are obtained using the iteration formula (41) and the solution of $u(x.t)$ in closed form is given by .Recall that $u(x.y.t) = \lim_{n\to\infty} u_n(x.y.t)$

Consequently, the exact solutions

$$u(x.y.t) = \frac{x}{1-t} \quad (46)$$

4. Conclusion

There are two main goals that we aimed for this work. The first is employing the powerful Reconstruction of the variational iteration to investigate Cauchy problems and the second one is showing the power of this method and its significant features. The two goals are achieved.

It is obvious that the method gives rapidly convergent successive approximations without any restrictive assumptions or transformation that may change the physical behavior of the problem. Reconstruction of the variational iteration gives several successive approximations through using the RVIM's iteration relation. Moreover, the RVIM reduces the size of calculations by not requiring the tedious Adomian polynomials, and hence the iteration is direct and straightforward. The RVIM uses the initial values for selecting the zeroth approximation, and boundary conditions, when given for bounded domains, can be used for justification only.

A clear conclusion can be drawn from the results that Reconstruction of the variational iteration method provides us an efficient tool to obtaining exact solutions of partial differential equations

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References

1. Liao, S. J. 2012. Homotopy analysis method in nonlinear differential equations. Higher Education Press, Springer, New York.
2. He, J. H. 1999. Variational iteration method–a kind of non-linear analytical technique: some examples. *International Journal of Non-Linear Mechanics* 34(4): 699-708.
3. He, J. H. 2000. Variational iteration method for autonomous ordinary differential systems. *Applied Mathematics and Computation* 114(2): 115-123.
4. He, J. H. 2007. Variational iteration method- Some recent results and new interpretations. *Journal of computational and applied mathematics* 207: 3-17.
5. Hesameddini, E., H. Latifizadeh. 2009. Reconstruction of Variational Iteration Algorithms

using the Laplace Transform, *Int. J. Nonlinear Sci. Numer. Simul.* 10:1377-1382.
6. McOwen, R. C. 1996. Partial Differential Equations: Methods and Applications, Prentice Hall, Inc.
7. Nikkar, A. Z., S. M. Mighani, S. B. Saghebian, Nojabaei, M. Daie. 2012. Development and Validation of an Analytical Method to the Solution of Modelling the Pollution of a System of Lakes. *Research Journal of Applied Sciences, Engineering and Technology* 5(1): 296-302.
8. Nikkar, A. Z. 2012. A New Approach for Solving Gas Dynamic Equation. *Acta Technica Corviniensis-Bulletin of Engineering* 4: 113-116.
9. Ghasempour, S. J., A. Vahidi, A. Z. Nikkar, S. M. Mighani. 2012. Analytical Approach to Some Highly Nonlinear Equations by Means of the RVIM. *Research Journal of Applied Sciences, Engineering and Technology* 5(1): 339-345.
10. Wazwaz, A. M. 2007. The variational iteration method: A powerful scheme for handling Linear and nonlinear diffusion equations, Computers and Mathematics with Applications. *Journal of Applied Mathematics* 54: 933-939.
11. Tveito, A., R. Winther. 2005. Introduction to Partial Differential Equations. Springer-Verlag, Berlin, Heidelberg.