



# Fundamental Topological Algebras

Mozhgan talimian\*

Department of Mathematics, Astara Branch, Islamic Azad University, Astara, Iran.

Received: 02 October 2017

Accepted: 13 November 2017

Published: 01 December 2017

## Abstract

A class of topological algebras, which we call it a fundamental one, has already been introduced. On locally multiplicative fundamental algebras, with a property similar to the normed case, we have before proved some theorems. Here we continue this process and get some results.

**Keywords:** Fundamental Topological Algebra; FLM algebra

## How to cite the article:

M. talimian, *Fundamental Topological Algebras*, Medbiotech J. 2017; 1(4): 175-177, DOI: 10.22034/mbt.2017.87080

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## Introduction

In (Ansari-Piri, 1990), the author introduces fundamental topological algebras [1], and in (Ansari-Piri, 2001) some basic theorems are proved on fundamental topological linear spaces [2]. In (Ansari-Piri, 2004), complete metrizable locally multiplicative fundamental topological algebras with a property similar to the case of Banach algebras have been introduced extending some theorems [3]. Here in this note, we continue this process. At first in Section 2, we recall some definitions and related results.

## Definitions and related results

**Definition 1:** A topological linear space  $A$  is said to be fundamental one if there exists  $b > 1$  such that for every sequence  $(x_n)$  of  $A$ , the convergence of  $b^n(x_n - x_{n-1})$  to zero in  $A$  implies that  $(x_n)$  is Cauchy.

**Proposition 2:** Let  $A$  be a fundamental topological linear space. Then for every  $c > 1$  and every sequence  $(x_n)$  of  $A$ , the convergence of  $c^n(x_n - x_{n-1}) \rightarrow 0$  in  $A$  implies that  $(x_n)$  is Cauchy [1-2,4].

**Definition 3:** A fundamental topological algebra is an algebra whose underlying topological linear space is fundamental.

**Theorem 4:** Let  $A$  be a complete metrizable fundamental topological algebra with unit element  $e$  and  $x \in A$ . If for some  $b > 1$ ,  $b^n x^n \rightarrow 0$  in  $A$ , then  $e - x \in \text{Inv}(A)$  [3,4.1].

**Definition 5:** A fundamental topological algebra is called to be locally multiplicative, if there exists a neighborhood  $U_0$  of zero such that for every neighborhood  $V$  of zero, the sufficiently large powers of  $U_0$  lie in  $V$ . We call such an algebra, an *FLM* algebra.

## Some results on fundamental topological algebras

In this section we state and prove some basic results on fundamental topological algebras.

**Theorem 1:** Let  $A$  be a complete metrizable fundamental topological algebra, and  $x \in A$ . If for some  $b > 1$ ,  $b^n x^n \rightarrow 0$  in  $A$ , then:

- $x$  is quasi-invertible and  $x^0 = -\sum_{n=1}^{\infty} x^n$ ,
- If  $A$  possesses a unit element, then  $1 - x$  is invertible and  $(1 - x)^{-1} = 1 + \sum_{n=1}^{\infty} x^n$ .

**Proof:** Suppose  $s_n = \sum_{k=1}^n x^k$ . Since  $b^n(s_n - s_{n-1}) = b^n x^n$ ,  $(s_n)$  is a Cauchy sequence. Let  $= \sum_{k=1}^{\infty} x^k = \lim_{n \rightarrow \infty} s_n$ . Now:

\* Corresponding Author Email: mozhgantelimian@yahoo.com

$$(1-x)(1+s_n) = (1-x) + \sum_{k=1}^n (x^k - x^{k+1}) = (1-x) + x - x^{n+1} \rightarrow 1.$$

Since the multiplication is continuous on  $A$ , we have:

$$(1-x)(1+y) = (1+y)(1-x) = 1.$$

**Corollary:** Let  $A$  be with the conditions of the theorem 3 and  $> 1$ . If  $b^n(1-x)^n \rightarrow 0$  in  $A$  then  $x$  is invertible.

Now, we have some results in a class of fundamental topological algebras. Although every locally bounded topological algebra satisfies in the condition of this class, but until now the author has no example of a locally convex one.

**Theorem 2:** Let  $A$  be a complete metrizable FLM algebra with a unit element. Then the set of all invertible elements of  $A$ , is an open subset of  $A$ .

**Proof:** Let  $x_0 \in \text{Inv}(A)$  and  $U_0$  be a neighborhood of zero which satisfies in 2 and  $> 1$ . Choose a neighborhood  $V_0$  such that  $x_0^{-1}V_0 \subseteq b^{-1}U_0$ . Now let  $y \in x_0 - V_0$ . Then  $x_0 - y \in V_0$  and  $1 - x_0^{-1}y \in x_0^{-1}V_0$ . Since:

$b(1 - x_0^{-1}y) \in bx_0^{-1}V_0 \subseteq U_0$ , We have:  $b^n(1 - x_0^{-1}y)^n \in U_0^n$  and  $b^n(1 - x_0^{-1}y)^n \rightarrow 0$  and so,  $x_0^{-1}y \in \text{Inv}(A)$ . Since  $y = x_0(x_0^{-1}y)$  thus  $y \in \text{Inv}(A)$ .

**Theorem 3:** Let  $A$  be a complete metrizable FLM algebra with a unit element and Let  $a \in A$ . Then, the  $sp(a)$  is compact.

**Proof:** Let  $a \in A$ . Define  $\phi: C \rightarrow A$  by  $\phi(\lambda) = \lambda - a$  then  $\phi$  is continuous and  $\phi^{-1}(\text{Inv}(A))$  is open. Since  $C \setminus \phi^{-1}(\text{Inv}(A)) = \phi^{-1}(\text{Sing}(A)) = sp(a)$  so  $sp(a)$  is closed.

Suppose  $U_0$  be a neighborhood of zero which satisfies in 2, and let  $\alpha \in C \setminus \{0\}$  such that  $a \in \alpha U_0$ , then  $(\alpha^{-1}a)^n \rightarrow 0$ . Suppose  $|\lambda| > 1$  and take  $b$  such that  $|\lambda| > b > 1$ , then  $(\frac{b}{\lambda})^n \rightarrow 0$  and  $b^n(\frac{a}{\alpha\lambda})^n \rightarrow 0$ . Therefore:

$1 - \frac{a}{\alpha\lambda} \in \text{Inv}(A)$  and  $\alpha\lambda \notin sp(a)$ . Now if  $\beta \in sp(a)$ , put  $\lambda = \frac{\beta}{\alpha}$  and then,  $|\lambda| \leq 1$  i.e.  $|\beta| \leq |\alpha|$ , thus the  $sp(a)$  is bounded (Ansari-Piri, 2004).

**Theorem 4:** Let  $A$  be a complete metrizable FLM algebra. Then, every multiplicative linear functional is continuous.

**Proof:** Let  $\phi: A \rightarrow C$  be a non-zero multiplicative linear functional and  $> 1$ . Suppose  $x \in A$  with  $b^n x^n \rightarrow 0$ . Put  $S_n = \sum_{k=1}^n x^k$ , then  $y = \sum_{k=1}^{\infty} x^k \in A$ . Since:  $y - xy = \lim S_n - x \lim S_n = \lim(S_n - xS_n) = x$ ,

We have  $\phi(x) \neq 1$ . If  $|\phi(x)| > 1$ , take  $x_0 = \frac{x}{\phi(x)}$ .

Since:

$$b^n x_0^n = b^n \frac{x^n}{\phi(x)^n} = \frac{1}{\phi(x)^n} b^n x^n \rightarrow 0$$

Thus we must have  $\phi(x_0) \neq 1$ , which is impossible; therefore if  $b^n x^n \rightarrow 0$  then  $|\phi(x)| < 1$ .

Now, let  $(x_n)$  be any null sequence in  $A$  and let  $U_0$  satisfies in 2 For  $\varepsilon > 0$ , take  $n \in Z^+$  so large that  $b^{-1}\varepsilon x_n \in U_0$ . Fix this  $n$  and suppose  $V$  be any neighborhood of zero. There exists  $K_0 \in Z^+$  such that  $k \geq K_0$  implies that  $U_0^k \subseteq V$  and so  $b^k(\varepsilon^{-k}x_n^k) \in V$  thus  $\lim_{k \rightarrow \infty} b^k(\varepsilon^{-k}x_n^k) = 0$  and so  $|\phi(\varepsilon^{-1}x_n)| < 1$ ; i.e.  $|\phi(x_n)| < \varepsilon$ , which says  $\lim_{x \rightarrow 0} \phi(x) = 0$ .

**Theorem 5:** Let  $A$  be a complete metrizable FLM algebra with unit element, and  $\phi$  a linear functional on  $A$  such that  $\phi(1) = 1$  and  $\ker \phi \subseteq \text{Sing}(A)$ . Then  $\phi$  is continuous.

**Proof:** Let  $b > 1$  and  $x \in A$  such that  $b^n x^n \rightarrow 0$  then  $1 - x \notin \text{Sing}(A)$  and then  $\phi(x) \neq \phi(1) = 1$ . If  $|\phi(x)| > 1$ , take  $x_0 = \frac{1}{\phi(x)}x$ .

Since:

$$b^n x_0^n = (\frac{1}{\phi(x)})^n b^n x^n \rightarrow 0.$$

So  $1 - x \notin \text{Sing}(A)$  and thus  $\phi(x_0) \neq 1$  which is impossible.

Now suppose  $U_0$  be as in 2.5,  $x_n \rightarrow 0$ ; and  $\varepsilon > 0$ . There exists  $N \in Z^+$  such that  $n \geq N$  implies that  $b\varepsilon^{-1}x_n \in U_0$  and hence,  $\lim_{k \rightarrow \infty} b^k(\varepsilon^{-k}x_n^k) = 0$ .

Therefore  $|\phi(\varepsilon^{-1}x_n)| < 1$ , i.e.  $\phi$  is continuous.

## Results

In (Ansari-Piri, 2004), we have extended some basic theorems from Banach algebras to complete metrizable fundamental and locally multiplicative fundamental topological algebras [3]. In this section we state and develop some other well-known results from Banach algebras to FLM algebras. We begin with a result about ideals. The following lemma with an algebraic proof is valid in any topological algebra.

**Lemma 1:** Let  $A$  be a topological algebra and  $J \subseteq A$  be a proper ideal and  $u$  be a right modular unit for  $J$ . Then  $u \notin J$  [4].

Now, we have the following results:

**Lemma 2:** Let  $J$  be a proper left ideal of a complete fundamental topological algebra, and  $u$  be a right modular unit for  $J$ . Then for no  $x \in J$  and no  $b > 1$ ;  $b^n(u - x)^n \rightarrow 0$ .

**Proof:** Let for some  $b > 1$  and  $x \in J$ ;  $b^n(u - x)^n \rightarrow 0$ . Then  $(u - x)$  is quasi-invertible [1,3-4]. Suppose  $(u - x) + y - y(u - x) = 0$ , then:

$$u = x - y + y(u - x) = x - yx - y(1 - u) \in J + J + J \subseteq J.$$

**Theorem 3:** Suppose  $A$  is an *FLM* algebra,  $J$  is a proper left ideal, and  $u$  is a right modular unit for  $J$ . Then  $u \notin \bar{J}$ .

**Proof:** Suppose  $b > 1$ ,  $U_0$  satisfies in Definition 2 and let  $u \in \bar{J}$ . Then, there exists an  $x \in J$  such that  $u - x \in b^{-1}U_0$  and  $b^n(u - x)^n \in U_0^n$ , and therefore,  $b^n(u - x)^n \rightarrow 0$  which is impossible.

We prove that in a complete metrizable *FLM* algebra, the set  $Inv(A)$  is an open subset of  $A$ , and the spectrum of any element is compact. In the next theorem we show that the map  $\phi: C - sp(a) \rightarrow A$  with  $\phi(\lambda) = (\lambda - a)^{-1}$  is holomorphic.

**Theorem 4:** Let  $A$  be a complete metrizable *FLM* algebra and  $a \in A$ . Then the map  $\phi: C - sp(a) \rightarrow A$  with  $\phi(\lambda) = (\lambda - a)^{-1}$  is holomorphic.

**Proof:** Suppose  $z_0 \in C - sp(a)$ . Then  $z_0 - a \in Inv(A)$ . Now, let  $b > 1$  and  $U_0$  be as in Definition 2 choose the balanced neighborhood  $V_0 \subseteq U_0$  and  $0 < \alpha < 1$  such that:  $\alpha(z_0 - a)^{-1} \in b^{-1}V_0$ . If  $|\gamma| < \alpha$ , then:

$\gamma(z_0 - a)^{-1} \in b^{-1}V_0 \subseteq b^{-1}U_0$ , and then  $b^n\gamma^n(z_0 - a)^{-n} \rightarrow 0$ ; and therefore  $1 - \gamma(z_0 - a)^{-1} \in Inv(A)$ .

Now, let  $|\eta - z_0| < \alpha$ . Then:

$$\eta - a = (z_0 - a)(1 - (\eta - z_0)(z_0 - a)^{-1}) \in Inv(A);$$

And by Ansari-Piri, 2004; Bonsall, 1973; Ansari-Piri, 1990.

$$(\eta - a)^{-1} = (z_0 - a)^{-1} + \sum_{n=1}^{\infty} (\eta - z_0)^n (z_0 - a)^{-n-1} .$$

Thus, for every  $f \in \hat{A}$ ,  $f \circ \phi$  is holomorphic on  $C - sp(a)$ .

We close the section with two more results on the sequences of  $Inv(A)$ , where the first one extends

Proposition 1:5.6 of (Dixon, 1981) with a weaker hypothesis [5].

**Theorem 5:** Suppose  $A$  is a complete metrizable *FLM* algebra with the unit element  $e$ . If  $(a_n)$  is a sequences of  $Inv(A)$  such that  $a_n \rightarrow a$  and for all  $n \in N$ ,  $a_n a = a a_n$ ; and moreover, the set  $\{a_n^{-1} : n \in N\}$  is bounded, then  $a \in Inv(A)$ .

**Proof:** Suppose  $U_0$  is as in Definition 2 Then there exists  $s > 1$  such that  $\forall n. a_n^{-1} \in sU_0$ . Take  $c > s$  and put  $c = cs^{-1} > 1$ . Since  $a_n \rightarrow a. \exists k$  such that  $a_k - a \in c^{-1}U_0$ . Then  $c^n s^{-n} (a_k - a)^n a_k^{-n} \rightarrow 0$  as  $n \rightarrow \infty$  i.e:  $b^n (e - a_k^{-1} a)^n \rightarrow 0$ . Therefore  $a \in Inv(A)$ , since by (Ansari-Piri, 2004; Bonsall, 1973; Ansari-Piri, 1990),  $a_k^{-1} a \in Inv(A)$ .

**Corollary:** In a complete metrizable *FLM* algebra  $A$ , if  $a \in \partial Inv(A)$ . and  $a_n \rightarrow a$  with  $a_n a = a a_n$ , then the set  $\{a_n^{-1} : n \in N\}$  is unbounded.

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