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Research Paper

Fundamental Topological Algebras

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Abstract

A class of topological algebras, which we call it a fundamental one, has already been introduced . On locally multiplicative fundamental algebras, with a property similar to the normed case, we have before proved some theorems. Here we continue this process and get some results.

Keywords*:* Fundamental Topological Algebra; FLM algebra

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Introduction

In (Ansari-Piri, 1990), the author introduces fundamental topological algebras [1], and in (Ansari-Piri, 2001) some basic theorems are proved on fundamental topological linear spaces [2]. In (Ansari-Piri, 2004), complete metrizable locally multiplicative fundamental topological algebras with a property similar to the case of Banach algebras have been introduced extending some theorems [3]. Here in this note, we continue this process. At first in Section 2, we recall some definitions and related results.

Definitions and related results

Definition1: A topological linear space A is said to be fundamental one if there exists $b > 1$ such that for every sequence (x_n) of A, the convergence of $b^n(x_n - x_{n-1})$ to zero in A implies that (x_n) is Cauchy.

Proposition 2: Let A be a fundamental topological linear space. Then for every $c > 1$ and every sequence (x_n) of A, the convergence of $c^n(x_n - x_{n-1}) \to 0$ in A implies that (x_n) is Cauchy [1-2,4].

Definition 3: A fundamental topological algebra is an algebra whose underlying topological linear space is fundamental.

Theorem 4: Let A be a complete metrizable fundamental topological algebra with unit element e and $x \in A$. If for some $b > 1$, $b^n x^n \to 0$ in A , then $e - x \in Inv(A)$ [3,4.1] .

Definition 5: A fundamental topological algebra is called to be locally multiplicative, if there exists a neighborhood U_0 of zero such that for every neighborhood V of zero, the sufficiently large powers of U_0 lie in V. We call such an algebra, an FLM algebra.

Some results on fundamental topological algebras

In this section we state and prove some basic results on fundamental topological algebras .

Theorem 1: Let A be a complete metrizable fundamental topological algebra, and $x \in A$. If for some $b > 1$, $b^n x^n \to 0$ in A , then:

a) x is quasi-invertible and $x^0 = -\sum_{n=1}^{\infty} x^n$,
b) If A possesses a unit element , then $1 - x$ is invertible and
$(1 - x)^{-1} = 1 + \sum_{n=1}^{\infty} x^n$.

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Proof: Suppose $s_n = \sum_{k=1}^{n} x^k$. Since $b^n(s_n - s_{n-1}) = b^n x^n$, (s_n) is a cauchy sequence. Let $\sum_{k=1}^{\infty} x^k = \lim_{n\to\infty} s_n$. Now:

$$(1-x)(1+s_n) = (1-x) + \sum_{k=1}^{n}(x^k - x^{k+1}) = (1-x) + x - x^{n+1} \to 1 .$$

Since the multiplication is continuous on A, we have:

$(1-x)(1+y) = (1+y)(1-x) = 1.$

Corollary: Let A be with the conditions of the theorem 3 and > 1. If $b^n(1-x)^n \to 0$ in A then x is invertible.

Now, we have some results in a class of fundamental topological algebras. Although every locally bounded topological algebra satisfies in the condition of this class, but until know the author has no example of a locally convex one.

Theorem 2: Let A be a complete metrizable FLM algebra with a unit element. Then the set of all invertible elements of A, is an open subset of A.

Proof: Let $x_0 \in Inv(A)$ and U_0 be a neighborhood of zero which satisfies in 2 and > 1. Choose a neighborhood V_0 such that ${x_0}^{-1}V_0 \subseteq b^{-1}U_0$. Now let $\in x_0 - V_0$. Then $x_0 - y \in V_0$ and $1 - {x_0}^{-1}y \in {x_0}^{-1}V_0$. Since:

$b(1 - {x_0}^{-1}y) \in b{x_0}^{-1}V_0 \subseteq U_0$, We have: $b^n(1 - {x_0}^{-1}y)^n \in {U_0}^n$ and $b^n(1 - {x_0}^{-1}y)^n \longrightarrow 0$ and so, ${x_0}^{-1}y \in Inv(A)$. Since $y = x_0({x_0}^{-1}y)$ thus $y \in Inv(A)$.

Theorem 3: Let A be a complete metrizable FLM algebra with a unit element and Let $\in A$. Then, the $sp(a)$ is compact.

Proof: Let $\in A$. Define $\phi: \mathbb{C} \longrightarrow A$ by $\phi(\lambda) = \lambda - a$ then ϕ is continuous and $\phi^{-1}(Inv(A))$ is open. Since $\mathbb{C} \setminus \phi^{-1}(Inv(A)) = \phi^{-1}(Sing(A)) = sp(a)$ so $sp(a)$ is closed.

Suppose U_0 be a neighborhood of zero which satisfies in 2, and let $\alpha \in \mathbb{C} \setminus \{0\}$ such that $a \in \alpha U_0$, then $(\alpha^{-1}a)^n \longrightarrow 0$. Suppose $|\lambda| > 1$ and take b such that $|\lambda| > b > 1$, than $(\frac{b}{\lambda})^n \longrightarrow 0$ and $b^n(\frac{a}{\alpha\lambda})^n \longrightarrow 0$. Therefore:

$1 - \frac{a}{\alpha\lambda} \epsilon Inv(A)$ and$\alpha\lambda \notin sp(a)$. Now if $\beta \in sp(a)$, put $\lambda = \frac{\beta}{\alpha}$ and then, $|\lambda| \leq 1$ i.e: $|\beta| \leq |\alpha|$, thus the $sp(a)$ is bounded (Ansari-Piri, 2004).

Theorem 4: Let A be a complete metrizable FLM algebra. Then, every multiplicative linear functional is continuous.

Proof: Let $\phi: A \to \mathbb{C}$ be a non-zero multiplicative linear functional and > 1. Suppose $x \in A$ with $b^n x^n \to 0$. Put $S_n = \sum_{k=1}^{n} x^k$, then $y = \sum_{k=1}^{\infty} x^k \in A$. Since: $y - xy = \lim S_n - x\lim S_n = \lim(S_n - xS_n) = x$,

We have $\phi(x) \neq 1$. If $|\phi(x)| > 1$, take $x_0 = \frac{x}{\phi(x)}$. Since:

$$b^n {x_0}^n = b^n \frac{x^n}{\phi(x)^n} = \frac{1}{\phi(x)^n} b^n x^n \to 0$$

Thus we must have $\phi(x_0) \neq 1$, which is impossible; therefore if $b^n x^n \to 0$ then $|\phi(x)| < 1$.

Now, let (x_n) be any null sequence in A and let U_0 satisfies in 2 For $\varepsilon > 0$, take $n \in Z^+$ so large that $b^{-1}\varepsilon x_n \in U_0$. Fix this n and suppose V be any neighborhood of zero. There exists $K_0 \in Z^+$ such that $k \geq K_0$ implies that $U_0^k \subseteq V$ and so $b^k(\varepsilon^{-k}x_n^k) \in V$ thus $\lim_{k\to\infty} b^k(\varepsilon^{-1}x_n)^k = 0$ and so ,$|\phi(\varepsilon^{-1}x_n)| < 1$; i.e. $|\phi(x_n)| < \varepsilon$, which says $\lim_{x\to 0}\phi(x) = 0$.

Theorem 5: Let A be a complete metrizable FLM algebra with unit element, and ϕ a linear functional on A such that $\phi(1) = 1$ and , $ker\phi \subseteq Sing(A)$. Then ϕ is continuous.

Proof: Let $b > 1$ and $x \in A$ such that $b^n x^n \to 0$ then $1 - x \notin Sing(A)$ and then $\phi(x) \neq \phi(1) = 1$. If $|\phi(x)| > 1$, take $x_0 = \frac{1}{\phi(x)}x$.

Since:

$$b^n {x_0}^n = (\frac{1}{\phi(x)})^n b^n x^n \to 0 .$$

So $1 - x \notin Sing(A)$ and thus $\phi(x_0) \neq 1$ which is impossible .

Now suppose U_0 be as in 2.5, $x_n \to 0$; and $\varepsilon > 0$. There exists $N \in Z^+$ such that $n \geq N$ implies that $b\varepsilon^{-1}x_n \in U_0$ and hence, $\lim_{k\to\infty} b^k(\varepsilon^{-k}x_n^k) = 0$.

Therefore $|\phi(\varepsilon^{-1}x_n)| < 1$, i.e: ϕ is continuous.

Results

In (Ansari-Piri, 2004), we have extended some basic theorems from Banach algebras to complete metrizable fundamental and locally multiplicative fundamental topological algebras [3]. In this section we state and develop some other well-known results from Banach algebras to FLM algebras. We begin with a result about ideals. The following lemma with an algebraic proof is valid in any topological algebra.

Lemma 1: Let A be a topological algebra and $J \subseteq A$ be a proper ideal and u be a right modular unit for J. Then $u \notin J$ [4].

Now, we have the following results:

Lemma 2: Let Jbe a proper left ideal of a complete fundamental topological algebra, and u be a right modular unit for J. Then for no $x \in J$ and no $b > 1$; $b^n(u - x)^n \to 0$.

Proof: Let for some $b > 1$ and $x \in J$; $b^n(u-x)^n \to 0$. Then $(u-x)$ is quasi-invertible [1,3-4]. Suppose$(u-x)+y-y(u-x)=0$, then:

$$u = x - y + y(u-x) = x - yx - y(1-u) \in J + J + J \subseteq J .$$

Theorem 3: Suppose A is an FLM algebra , J is a proper left ideal, and u is a right modular unit for J. Then $u \notin \bar{J}$.

Proof: Suppose $b > 1$, U_0 satisfies in Definition 2 and let $u \in \bar{J}$.Then, there exists an $x \in J$ such that $u - x \in b^{-1}U_0$ and $b^n(u-x)^n \in U_0^n$, and therefore, $b^n(u-x)^n \to 0$ which is impossible.

We prove that in a complete metrizable FLM algebra, the set $Inv(A)$ is an open subset of A, and the spectrum of any element is compact. In the next theorem we show that the map $\phi: \mathbb{C} - sp(a) \to A$ with $\phi(\lambda) = (\lambda - a)^{-1}$is holomorphic.

Theorem 4: Let A be a complete metrizable FLM algebra and $\in A$. Then the map $\phi: \mathbb{C} - sp(a) \to A$ with $\phi(\lambda) = (\lambda - a)^{-1}$ is holomorphic.

Proof: Suppose $z_0 \in \mathbb{C} - sp(a)$. Then $z_0 - a \in Inv(A)$. Now, let $b > 1$ and U_0 be as in Definition 2 choose the balanced neighborhood $V_0 \subseteq U_0$ and $0 < \alpha < 1$ such that: $\alpha(z_0 - a)^{-1} \in b^{-1}V_0$. If $|\gamma| < \alpha$, then:

$\gamma(z_0 - a)^{-1} \in b^{-1}V_0 \subseteq b^{-1}U_0$, and then $b^n\gamma^n(z_0 - a)^{-n} \to 0$; and therefore $1 - \gamma(z_0 - a)^{-1} \in Inv(A)$.

Now, let $|\eta - z_0| < \alpha$. Then:

$$\eta - a = (z_0 - a)(1 - (\eta - z_0)(z_0 - a)^{-1}) \in Inv(A);$$

And by Ansari-Piri, 2004; Bonsall, 1973; Ansari-Piri, 1990.

$$(\eta - a)^{-1} = (z_0 - a)^{-1} + \sum_{n=1}^{\infty} (\eta - z_0)^n (z_0 - a)^{-n-1} \quad .$$

Thus, for every $f \in \acute{A}$, $f \circ \phi$ is holomorphic on $\mathbb{C} - sp(a)$.

We close the section with two more results on the sequences of $Inv(A)$, where the first one extends Proposition 1:5.6 of (Dixon, 1981) with a weaker hypothesis [5].

Theorem 5: Suppose A is a complete metrizable FLM algebra with the unit element e. If $(a_n)$is a sequences of $Inv(A)$ such that $a_n \to a$ and for all $n \in N$, $a_n a = a a_n$; and moreover, the set $\{a_n^{-1} : n \in N\}$ is bounded, then $a \in Inv(A)$.

Proof: SupposeU_0 is as in Definition 2 Then there exists $s > 1$ such that $\forall n.$ $a_n^{-1} \in sU_0$. Take $c > s$ and put $= cs^{-1} > 1$. Since $a_n \to a. \exists k$ such that $a_k - a \in c^{-1}U_0$. Then $c^n s^{-n}(a_k - a)^n a_k^{-n} \to 0$ as $n \to \infty$ i.e: $b^n(e - a_k^{-1}a)^n \to 0$. Therefore $a \in Inv(A)$, since by (Ansari-Piri, 2004; Bonsall, 1973; Ansari-Piri, 1990), $a_k^{-1}a \in Inv(A)$.

Corollary: In a complete metrizable FLM algebra A, if $a \in \partial Inv(A)$. and $a_n \to a$ with $a_n a = a a_n$,then the set $\{a_n^{-1} : n \in N\}$ is unbounded.

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